

# COUNTABLE SUCCESSOR ORDINALS AS GENERALIZED ORDERED TOPOLOGICAL SPACES

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ABSTRACT. We prove the following Main Theorem: Assume that any continuous image of a Hausdorff topological space  $X$  is a generalized ordered space. Then  $X$  is homeomorphic to a countable successor ordinal (with the order topology).

The converse trivially holds.

## 1. Introduction and Main Theorem

All topological spaces are assumed to be Hausdorff. Remind that  $L$  is a *Linearly Ordered Topological Space* (LOTS) whenever there is a linear ordering  $\leq^L$  on the set  $L$  such that a basis of the topology  $\lambda^L$  on  $L$  consists of all open convex subsets. A *convex* set  $C$  in a linear ordering  $M$  is a subset of  $M$  with the property: for every  $x < z < y$  in  $M$ , if  $x, y \in C$  then  $z \in C$ . The above topology, denoted by  $\lambda^L$  is called an *order topology*. Since the order  $\leq^L$  defines the topology  $\lambda^L$  on  $L$  (but not vice-versa), we denote also by  $\langle L, \leq^L \rangle$  the structure including the topology  $\lambda^L$ .

A topological space  $\langle X, \tau^X \rangle$  is called a *Generalized Ordered Space* (GO-space) whenever  $\langle X, \tau^X \rangle$  is homeomorphic to a subspace of a LOTS  $\langle L, \lambda^L \rangle$ , that is  $\tau^X = \lambda^L \upharpoonright X := \{U \cap X : U \in \lambda^L\}$  (see [2]).

Evidently, every LOTS, and thus any GO-space, is a Hausdorff topological space, but not necessarily separable or Lindelöf. The Sorgenfrey line  $Z$  is an example of a GO-space, which is not a LOTS, and such that every subspace of  $Z$  is separable and Lindelöf (see [4]).

By definition, every subspace of a GO-space is also a GO-space. In this article, in a less traditional manner, we say that a space  $X$  is a *hereditarily GO-space* if every continuous image of  $X$  (in particular,  $X$  itself) is a GO-space.

**Main Theorem 1.1.** *Every hereditarily GO-space is homeomorphic to a countable successor ordinal, considered as a LOTS. The converse obviously holds.*

This result is closely related to the following line of research: characterize Hausdorff topological spaces  $X$  such that all continuous images of  $X$  have the topological property  $\mathcal{P}$ . All questions listed below for concrete  $\mathcal{P}$  are still open.

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**Problem 1.2.**

- (1) Characterize Hausdorff spaces such that all continuous images of  $X$  are regular.
- (2) Characterize Hausdorff spaces  $X$  such that all continuous images of  $X$  are normal. (This was partially solved by W. Fleissner and R. Levy in [5, 6]).
- (3) Characterize Hausdorff spaces such that all continuous images of  $X$  are realcompact. (This question has been formulated in [1]).
- (4) Characterize Hausdorff spaces such that all continuous images of  $X$  are paracompact.
- (5) Characterize Hausdorff spaces such that all continuous images of  $X$  are monotonically normal. (This is related to “Nickel Conjecture” answered positively by M. E. Rudin [12]).  $\square$

**Remark 1.3.** Let  $\mathfrak{b}$  be the minimal cardinality of unbounded subsets of  $\omega^\omega$ . Recently M. Bekkali and S. Todorćević proved the following relevant result: Continuous zero-dimensional images of a compact LOTS of weight less than  $\mathfrak{b}$  is itself a LOTS [3, Theorem 4.2].

Note also a recent paper [14], which studies topological properties  $\mathcal{P}$  that are reflectable in small continuous images. For instance, a GO-space  $X$  is hereditarily Lindelöf iff all continuous images of  $X$  have countable pseudocharacter [14].

As a special case of Main Theorem 1.1, we obtain the following fact.

**Corollary 1.4.** *Assume that any continuous image of a Hausdorff topological space  $X$  is a LOTS. Then  $X$  is homeomorphic to a countable successor ordinal.*

In order to make this paper widely readable, we have tried to give self-contained and elementary proofs, even when our results could be deduced from more general theorems. By these reasons, and for the readers’ convenience, we include a separate and short proof of Corollary 1.4 in Section 2.

In our paper, a *GO-structure*, formally, is a 4-tuple  $\langle X, \tau^X, L, \leq^L \rangle$  where

- (1)  $X \subseteq L$ ,
- (2)  $\langle L, \leq^L \rangle$  is a linear ordering with the order topology  $\lambda^L$ , and
- (3) the order  $\leq^X$  on  $X$  is the restriction  $\leq^L \upharpoonright X$  of  $\leq^L$  to  $X$ , and  $\tau^X$  is the topology  $\lambda^L \upharpoonright X := \{U \cap X : U \in \lambda\}$

Any GO-space  $X$  can be written under the above structure. In [7], [10] GO-spaces are denoted by  $\langle X, \tau^X, \leq^X \rangle$  where  $\leq^X$  is the restriction of  $\leq^L$  to  $X$ . It is easy to see that the following are equivalent.

- (i)  $\langle X, \tau^X, \leq^X \rangle$  is a GO-space and
- (ii)  $\lambda^X \subseteq \tau^X$  and  $\tau^X$  has a base consisting of convex sets.

Many times we denote  $\langle X, \tau^X, L, \leq^L \rangle$  by  $\langle X, \tau^X, L, \lambda^L \rangle$ .

The proof of Main Theorem 1.1, is organized as follows. In §3, we present the basic facts on LOTS and GO-spaces. In particular, Proposition 3.4 shows that if  $\langle X, \tau^X, L, \leq^L \rangle$  is a GO-structure, then we may assume that  $L$  is a complete ordering and that  $X$  is topologically dense in  $\langle L, \lambda^L \rangle$ . In §4.1 we show that any hereditarily GO-space satisfies c.c.c. property: every family of pairwise disjoint nonempty open set is countable (Lemma 4.1). In §4.2 we prove that any hereditarily GO-space has no countable closed and relatively discrete subset (Corollary 4.5): we recall that a subset  $D$  of a space  $Y$  is *relatively discrete* whenever there is a family  $\mathcal{U}_D := \{U_d : d \in D\}$  of open subsets of  $Y$  such that  $D \cap U_d = \{d\}$  for every  $d$ .

Proposition 4.8 shows that a hereditarily GO-space is a subspace of a scattered linear order (with the order topology). Finally, in §4.4 we conclude the proof of Main Theorem.

## 2. Elementary proof of Main Theorem for LOTS: Corollary 1.4

We assume that the reader is familiar with the properties of LOTS and give a self-contained proof of Corollary 1.4. To prove the result, we need some preliminaries.

**Fact 1.** *Let  $\langle L, \leq^L \rangle$  be a LOTS. If  $\langle L, \leq^L \rangle$  is a scattered ordering then  $\langle L, \lambda^L \rangle$  is a scattered topological space.*  $\square$

The converse does not hold: consider the lexicographic sum  $N := \sum_{r \in \mathbb{Q}} \mathbb{Z}_r$  of copies  $\mathbb{Z}_r$  of the integers  $\mathbb{Z}$  over the rational chain  $\mathbb{Q}$ :  $\langle N, \leq^N \rangle$  is not a scattered ordering but  $\langle N, \lambda^N \rangle$  is a discrete space (this example will be used again in Part (1) of Remark 3.3).

The next fact is used implicitly in the arguments that follow.

**Fact 2.** *Let  $\langle L, \leq^L \rangle$  be a LOTS and  $F$  be a closed subspace of  $\langle L, \lambda^L \rangle$ . Then the induced topology  $\lambda^L|_F$  on  $F$  is the order topology on  $F$  defined by the restriction  $\leq^L|_F$  of  $\leq^L$  on  $F$ .*  $\square$

**Fact 3.** *Let  $\langle L, \leq^L \rangle$  be a scattered linear ordering then  $\langle L, \lambda^L \rangle$  is 0-dimensional (i.e.  $\langle L, \lambda^L \rangle$  has a base consisting of clopen sets).*

*Proof.* The proof uses the fact that if  $\langle L, \leq^L \rangle$  is a scattered linear ordering, then the Dedekind completion  $\langle L^c, \leq^{L^c} \rangle$  of  $\langle L, \leq^L \rangle$  is also a scattered linear ordering.  $\square$

**Fact 4.** *Let any continuous image of a Hausdorff topological space  $Y$  is a LOTS. Then  $Y$  satisfies c.c.c.. In particular,  $\omega_1$  and  $\omega_1^*$  are not order-embeddable in  $Y$ .*

**Fact 5.** *Let  $Y$  be a 0-dimensional LOTS. If  $D$  is a countable closed and discrete subset of  $Y$  then  $D$  is a continuous image of  $Y$ .*

*Proof.* Let  $\langle U_d \rangle_{d \in D}$  be clopen subsets of  $Y$  such that  $U_d \cap D = \{d\}$  for  $d \in D$ . Fix  $d_0 \in D$ . Let  $\approx$  be the equivalence relation on  $Y$  defined by  $x \approx y$  whenever there is  $d \in D \setminus \{d_0\}$  such that  $x, y \in U_d$ , or  $x, y \notin U := \bigcup \{U_d : d \in D \setminus \{d_0\}\}$ . Then  $Y/\approx$  is a continuous image of  $Y$ ,  $Y/\approx$  is Hausdorff and  $D$  is homeomorphic to  $Y/\approx$ .  $\square$

**Fact 6.** *Let  $D$  be a countable and discrete space. Then  $D$  has a continuous image which is not homeomorphic to a LOTS.*

*Proof.* Consider  $\omega$  as a discrete space. Let  $\mathcal{U}$  on  $\omega$  be a non-principal ultrafilter on  $\omega$  and let  $*$  be a new element with  $*$   $\notin \omega$ . We equip  $\mathbb{N}^* := \omega \cup \{*\}$  with the topology induced from the Stone–Čech compactification  $\beta\mathbb{N}$ . It is well-known that  $\mathbb{N}^*$  is not a LOTS: this is so because  $\mathbb{N}^*$  is separable and its topology does not have a countable base [4, §3.6]. Evidently, countable  $\mathbb{N}^*$  is a continuous image of  $\omega$ .  $\square$

As a consequence of Facts 4–6, we have the following result.

**Fact 7.** *Let  $\langle X, \leq^X, \lambda^X \rangle$  be a scattered linear ordering. Assume that any continuous image of  $X$  is a LOTS. Then the following holds*

- (1) *If  $x$  is in the topological closure of a nonempty subset  $A$  in  $X$ , then there is a countable monotone sequence of elements of  $A$  converging to  $x$ .*
- (2) *Every monotone sequence  $\langle x_n \rangle_{n \in \omega}$  converges. In particular, there exist both minimum and maximum in  $\langle X, \leq \rangle$ .*  $\square$

Now we are in a position to finish the argument.

*Proof of Main Theorem (for LOTS).* Let  $\equiv$  be the equivalence on  $\langle X, \leq^X, \lambda^X \rangle$  defined by:

$$x \equiv y \text{ if and only if } \begin{cases} x \leq y & \text{and } [x, y] \text{ is a scattered linear ordering} \\ x \leq y & \text{and } [y, x] \text{ is a scattered linear ordering} \end{cases}$$

Note that each  $\equiv$ -class is closed for the topology  $\lambda^X$  and each  $\equiv$ -class is convex and scattered for the order  $\leq^X$ . Moreover,  $X/\equiv$ , denoted by  $X_1$ , is a LOTS.

Note that there are no consecutive classes in  $X_1$ , and thus  $X_1$  is order-dense. Also the map  $\varphi : X \rightarrow X/\equiv$ , preserving supremum and infimum, is increasing and continuous.

Let  $N$  be a linear ordering and  $N^c$  its Dedekind completion. Recall that a *cut* is a member of  $N^c \setminus (L \cup \{\min(N^c), \max(N^c)\})$ . For instance, in the chain  $\mathbb{Q}$  of rationals, the cuts are the irrationals.

**Case 1.** *The set  $\Gamma (:= X_1^c \setminus X_1)$  of cuts of  $X_1$  has no consecutive elements and  $\Gamma$  is topologically dense in  $X_1$ .*

So,  $\Gamma$  is order-dense and  $\Gamma$  has no first and no last element. Note that, in that case,  $X_1$  is 0-dimensional. Let  $c \in \Gamma$ . Let  $\langle c_\alpha \rangle_{\alpha < \lambda}$  be a strictly increasing sequence cofinal in  $(-\infty, c)$  with  $\lambda$  regular. By Fact 4,  $\lambda = \omega$ . Since  $c \in \Gamma$ ,  $D := \{x_\alpha : \alpha \in \omega\}$  is a countable discrete and closed subset of  $X_1$ , which contradicts Fact 6. So, Case 1 does not occur.

**Case 2.** *The set  $\Gamma (:= X_1^c \setminus X_1)$  of cuts of  $X_1$  has two consecutive elements, or  $\Gamma$  is not topologically dense in  $X_1$ .*

Then there is a nonempty open interval  $(u, v)$  of  $X$  such that  $(u, v) \cap \Gamma = \emptyset$ . We set  $X'_1 := [u, v]$ . So  $X'_1$  is connected, infinite and order-dense. Also  $X'_1$  is a continuous image of  $X_1$ . Let  $X_2$  be the quotient space of  $X'_1$ , obtained by identification of  $u$  and  $v$ . Obviously,  $X_2$  is connected. Since for every  $t \in X_2$  the set  $X_2 \setminus \{t\}$  is connected,  $X_2$  is not a LOTS.

We have proved that  $|X_1| = 1$ , that is:  $X$  is a scattered linear ordering. By Fact 1,  $X$  is a scattered topological space. Moreover,  $X$  satisfies c.c.c., and thus  $\omega_1$  and  $\omega_1^*$  are not order-embeddable in the scattered linear ordering  $X$ . In particular, the space  $X$  has only countably many isolated points. Also the space  $X$  has no infinite and discrete subset. Hence the linear ordering  $X$  is complete and thus the space  $X$  is compact. We have proved that  $X$  is a countable compact and scattered space, that is,  $X$  is homeomorphic to  $\alpha + 1$  for some  $\alpha < \omega_1$ .  $\square$

### 3. Basic facts on LOTS and GO-spaces

Let  $S$  be a set,  $\mathcal{U} \subseteq \mathcal{P}(S)$  and  $T \subseteq S$ . We set  $\mathcal{U} \restriction T = \{U \cap T : U \in \mathcal{U}\}$ . Recall that a linear ordering  $\langle M, \leq^M \rangle$  is *complete* whenever every subset  $A$  of  $M$  has the supremum  $\sup^M(A)$  and the infimum  $\inf^M(A)$ . In particular, there exist both the maximum  $\max(M)$  and the minimum  $\min(M)$  in  $M$ .

Let  $N$  be a linear ordering. The *Dedekind completion* of  $N$ , denoted by  $N^c$ , is a complete linear ordering containing  $N$ , such that  $N^c$  is minimal with respect to this property. That is,

- (D1)  $N^c$  is a complete chain,
- (D2) for every  $x, y \in N$ : if  $x <^N y$  then  $x <^{N^c} y$ ,

- (D3) for every  $x \in N^c$  there are sets  $A, B \subseteq N$  such that  $x \notin A \cup B$  and  $\sup^{N^c}(A) = x = \inf^{N^c}(B)$ .

For instance,  $\min(N^c) = \sup^{N^c}(\emptyset) = \inf^{N^c}(N)$ . Notice also that the Dedekind completion  $N^c$  of  $N$  is unique up to an order-isomorphism.

In a linear ordering  $N$ , a *cut in  $N$*  is a member of  $N^c \setminus (N \cup \{\min(N^c), \max(N^c)\})$ , [11, 2.22, 2.23]. For example, the cuts for the chain  $\mathbb{Q}$  of rationals are irrationals. Next we say that  $\langle a, b \rangle$  are *consecutive in  $N$*  whenever  $a, b \in N$ ,  $a < b$  and  $(a, b)^N = \emptyset$ . So, for any linear ordering  $N$ :

- (1) if  $\langle a', b' \rangle$  are consecutive in  $N$  then  $\langle a', b' \rangle$  are consecutive in  $N^c$ , and
- (2) if  $\langle a', b' \rangle$  are consecutive in  $N^c$  then  $a', b' \in N$  and  $\langle a', b' \rangle$  are consecutive in  $N$ .

The following fact is well-known.

**Proposition 3.1.** *Let  $N$  be a LOTS.*

- (1)  $\langle N, \lambda^N \rangle$  is a compact space if and only if  $\langle N, \leq^N \rangle$  is order complete.
- (2)  $\langle N, \lambda^N \rangle$  is a connected space if and only if  $\langle N, \leq^N \rangle$  has no cuts and no consecutive elements.  $\square$

Next we introduce some basic notions on scattered linear orders and scattered spaces. Let  $N$  be a linear ordering. We say that  $N$  is *order-dense*, or *dense* if between two elements of  $N$  there is a member of  $N$ . Notice that for any dense linear order  $N$ , the rational chain  $\mathbb{Q}$  is order-embeddable in  $N$ . A linear order  $N$  is called *order-scattered* or simply *scattered*, whenever the rational chain  $\mathbb{Q}$  is not embeddable in  $N$ . For example,  $\omega_1$  and its converse ordering  $\omega_1^*$  are scattered linear orderings (by the definition,  $\langle \omega_1^*, \leq \rangle$  is the ordering  $\langle \omega_1, \geq \rangle$ ).

A space  $Y$  is *dense-in-itself* if  $Y$  is nonempty and has no isolated point. A dense-in-itself and closed subspace  $Y$  of a space  $Z$  is called a *perfect* subspace of  $Z$ . A space  $X$  is called *topologically-scattered*, or simply *scattered (space)*, whenever  $X$  does not contain a perfect subspace, that is, every nonempty subset  $A$  of  $X$  with the induced topology has an isolated point in  $A$ . We state other well-known facts about LOTS. For completeness we include the proofs.

**Proposition 3.2.** *Let  $N$  be a LOTS.*

- (1) *The following hold.*
  - (a) *If  $\langle N, \leq^N \rangle$  is a scattered linear ordering then  $\langle N, \lambda^N \rangle$  is topologically scattered.*
  - (b) *Assume that  $\langle N, \leq^N \rangle$  is a complete chain, i.e., by Proposition 3.1(1),  $\langle N, \lambda^N \rangle$  is a compact space. Then the following are equivalent.*
    - (i)  *$\langle N, \lambda^N \rangle$  is a scattered topological space.*
    - (ii)  *$\langle N, \leq^N \rangle$  is a scattered linear ordering.*
- (2) *If  $\langle N, \leq^N \rangle$  is order-scattered then  $\langle N, \lambda^N \rangle$  is 0-dimensional.*
- (3) *If  $N$  has only countably many isolated points then  $N$  is countable.*

*Proof.* (1) (a) Assume that  $\langle N, \lambda^N \rangle$  is not a scattered space. Let  $D \subseteq N$  be a dense-in-itself subset of  $N$ . Then  $\langle D, \leq^N \upharpoonright D \rangle$  contains an order-dense subset, and thus  $N$  is not a scattered chain.

(b) Suppose that  $\langle N, \lambda^N \rangle$  is compact and that  $\langle N, \leq^N \rangle$  is not order-scattered. Let  $S \subseteq N$  be an order-dense subset of  $N$ , and let  $T$  be its topological closure in  $\langle N, \lambda^N \rangle$ . Then  $T$  has no isolated points, i.e.  $T$  is dense-in-itself. By compactness,  $T$  is compact and thus  $T$  is perfect. Hence  $\langle N, \lambda^N \rangle$  is not a scattered space.

We prove a little bit more. We have  $N = N^c$ . Let  $M$  be a linear ordering and let  $M^c$  be its the Dedekind completion. The following are equivalent: (i)  $M$  is a scattered chain, (ii)  $\mathbb{Q}$  is not order-embeddable in  $M$ , (iii)  $\mathbb{Q}$  is not order-embeddable in  $M^c$ , and (iv)  $M^c$  is a scattered chain. Now since  $\langle M^c, \leq^{M^c} \rangle$  is a complete chain,  $\langle M^c, \lambda^{M^c} \rangle$  is a compact space. Therefore the previous items are equivalent to each of the following (v)  $\mathbb{Q}$  is not order-embeddable in  $M^c$ , and (vi)  $\langle M^c, \lambda^{M^c} \rangle$  is a scattered space.

(2) In the proof of Part (1), we have seen that if  $N$  is a scattered chain, then its Dedekind completion  $N^c$  is a scattered chain and thus  $N^c$ , considered as a LOTS, is compact and topologically-scattered. Therefore  $N^c$  is 0-dimensional, and thus  $N$  is also 0-dimensional.

(3) By the hypothesis, the set  $S := \text{Iso}(N)$  of isolated points in  $N$  is countable. Since  $N$  is a scattered space,  $S$  is topologically dense in  $N$ . Since  $S$  is a chain, by the proof of Part (1), the chain  $S^c$  is scattered. We claim that  $S^c$  is countable. This is so, because if  $S^c$  is an uncountable scattered chain, then  $\omega_1$  or  $\omega_1^*$  is order-embeddable in  $S^c$  and thus the same holds for  $S$ , that is,  $S$  is uncountable contradicting our assumptions.

Next, since  $S^c$  is countable,  $N$  must be countable. Indeed, there exists a continuous (increasing) map  $f$  from  $N$  into  $S^c$  such that  $|f^{-1}(x)| \leq 2$  for any  $x \in N$ . (That is, the case if  $N := \omega + 1 + 1 + \omega^*$ , and thus  $S := \text{Iso}(N) = \omega + \omega^*$  and  $S^c = \omega + 1 + \omega^*$ .)  $\square$

**Remark 3.3.** (1) Proposition 3.2(1)(a) is not reversible. As an example, consider the lexicographic sum  $N := \sum_{q \in \mathbb{Q}} \mathbb{Z}_q$  of copies  $\mathbb{Z}_q$  of (the the chain of integers)  $\mathbb{Z}$ , indexed by the chain of rationals  $\mathbb{Q}$ . Then  $N$  is a non-scattered linear ordering, but  $N$  is a topological discrete LOTS and thus  $N$  is a scattered topological space.

(2) Recall that the Dedekind completion of  $M := (0, 1) \cap \mathbb{Q}$  is  $M^c = [0, 1]^{\mathbb{R}}$ . On the other hand,  $M$  is topologically dense in the Cantor set  $2^\omega$  (considered as a subset of  $\mathbb{R}$ ).

(3) Let  $X := \{1/n : n > 0\} \cup \{-1\} \subset \mathbb{R} := L$ . Then  $\langle X, \lambda^X \rangle$  is compact, but  $\langle X, \tau^X \rangle$  is infinite and discrete.  $\square$

- Let  $\langle X, \tau^X, L, \leq^L \rangle$  be a GO-space. Then  $\lambda^L \upharpoonright X \subseteq \tau^X$ .

Indeed, let  $a < b$  in  $X$ . So  $(a, b)^X \in \lambda^X$  and thus, by the definition,  $(a, b)^X = (a, b)^N \cap X \in \tau^X$ .

The following result is well-known. For completeness we include its proof.

**Proposition 3.4.** *Let  $\langle X, \tau^X \rangle$  be a GO-space. Let  $\langle L, \leq^L \rangle$  be such that  $\langle X, \tau^X, L, \leq^L \rangle$  is a GO-space. Without loss of generality we may assume that  $L$  satisfies:*

- (H1)  $X$  is topologically dense in  $\langle L, \lambda^L \rangle$ ;
- (H2)  $\langle L, \leq^L \rangle$  is a complete linear ordering.

*Proof.* The proof follows from the following two facts.

*Fact 1.* *Let  $\langle N, \leq^N \rangle$  be a linear ordering and  $\langle N^c, \leq^{N^c} \rangle$  be its Dedekind completion. Then  $\lambda^N = \tau^{N^c} \upharpoonright N$ . That is, the order topology  $\lambda^N$  is the induced topology of  $\tau^{N^c}$  on  $N$ .*

*Proof.* Let  $c \in N^c$ . Then  $c$  is a cut if and only if  $c \notin N \cup \{\min(N), \max(N)\}$  and  $c$  has no predecessor nor a successor in  $N^c$ . Obviously,  $\lambda^N \subseteq \tau^{N^c} \upharpoonright N$ . Next let  $(a, b)^{N^c} \in \lambda^{N^c}$  be an open convex set in  $\langle N^c, \leq^{N^c} \rangle$ . So  $a, b \in N^c$ . If  $a \notin N$  then  $a = \inf(\{a' \in N : a' > a\})$  and if  $b \notin N$  then  $b = \inf(\{b' \in N : b' < b\})$ . So  $(a, b)^{N^c} \cap N$  is an union of open convex sets  $(a', b')^N$  where  $a', b' \in N$ .  $\square$

*Fact 2.* Let  $\langle X, \tau^X, N, \leq^N \rangle$  be a GO-structure such that  $\langle N, \tau^N \rangle$  is a complete ordering. Let  $L$  be the topological closure of  $X$  in the space  $\langle N, \lambda^N \rangle$ . Then  $\tau^X = \lambda^L \upharpoonright X$ . That is,  $\langle X, \tau^X, L, \leq^N \upharpoonright L \rangle$  is a GO-structure.

*Proof.* It suffices to show that for every  $a < b$  in  $N$  there are  $a' < b'$  in  $L$  such that  $(*)$ :  $(a', b')^L \cap X = (a, b)^N \cap X$ . Fix  $a < b$  in  $N$ . If  $a \in X$  ( $b \in X$ ) set  $a' = a$  ( $b' = b$ ). Next suppose that  $a \in N \setminus L$ . Since  $N$  is a complete ordering and  $N \setminus L$  is open in  $L$ , there is a (maximal) open convex set  $(\alpha, \alpha')^N$  in  $N$  such that  $a \in (\alpha, \alpha')^N$ ,  $(\alpha, \alpha')^N \cap L = \emptyset$  and  $\alpha, \alpha' \in L$ ; and we set  $a' = \alpha$ .

Similarly, suppose that  $b \in N \setminus L$ . Again, since  $N$  is complete and  $N \setminus L$  is open in  $L$ , there is a (maximal) open convex set  $(\beta, \beta')^N$  such that  $b \in (\beta, \beta')^N$ ,  $(\beta, \beta')^N \cap L = \emptyset$  and  $\beta, \beta' \in L$ ; and we set  $b' = \beta'$ . Now obviously  $(a', b')^L$  is as required in  $(*)$ .  $\square$

Now let  $\langle X, \tau^X, N, \leq^N \rangle$  be a GO-structure. By Fact 1 we may assume that  $N$  is a complete ordering. Finally, the result follows from Fact 2.  $\square$

**Remark 3.5.** (1) In general  $\lambda^X \subsetneq \tau^X$ . For example, consider  $L = \omega_1$  and let  $\text{Lim}$  be the set of all countable limit ordinals. We set  $X = \omega_1 \setminus \text{Lim}$ . We have:

- (i)  $\tau^X$  is the discrete topology,
- (ii)  $X$  is topologically dense in  $\langle \omega_1, \lambda^{\omega_1} \rangle$ , and
- (iii)  $X$  is order-isomorphic to  $\omega_1$  and thus  $\langle X, \lambda^{\omega_1} \rangle$  is homeomorphic to the ordinal space  $\omega_1$ .

(2) The one-point compactification of an uncountable discrete space is not a GO-space (by Lemma 4.1), and there is a countable space which is not a GO-space (by Lemma 4.4).  $\square$

For completeness we recall the proof of the following fact.

**Proposition 3.6.** [10, Lemma 6.1] Let  $\langle X, \tau^X, L, \leq^L \rangle$  be a GO-structure satisfying (H1) and (H2). So  $\langle X, \leq^L \upharpoonright X \rangle$  is a linear ordering.

- (1) If  $\langle X, \tau^X \rangle$  is a compact space then  $X = L$  and  $\tau^X = \lambda^X$ .
- (2) If  $\langle X, \tau^X \rangle$  is a connected space then  $\tau^X = \lambda^X$ .

*Proof.* (1) Since  $X$  is compact then  $X$  is closed in  $\langle L, \lambda^L \rangle$ . By (H1),  $X = L$  and thus  $\tau^X = \lambda^X$ .

(2) Next suppose that  $\langle X, \tau^X \rangle$  is connected. Then  $X$ , considered as the LOTS  $\langle X, \lambda^X \rangle$ , has no consecutive point and no cut because for each final subset of  $\langle X, \leq^L \upharpoonright X \rangle$  is closed in  $\langle X, \tau^X \rangle$ . Therefore,  $X \setminus \{\min(L), \max(L)\} = L \setminus \{\min(L), \max(L)\}$  and thus  $\tau^X = \lambda^X$ .  $\square$

#### 4. Proof of Main Theorem

As one of the main parts of Main Theorem 1.1 Proposition 4.8 implies that it suffices to assume that  $\langle L, \leq \rangle$  is a scattered linear order. To prove this result, we use Corollary 4.5 (in §4.2) which says that a GO-space does not contain an infinite countable relatively discrete closed subset.

Recall that  $\langle X, \tau \rangle$  is a GO-space means that  $\langle X, \tau, L, \leq \rangle$  is a GO-structure. So, we have  $X \subseteq L$ . Also for simplicity  $\langle X, \tau \rangle$  is denoted by  $X$ . In the sequel, by Proposition 3.4, we assume that the GO-structure  $\langle X, \tau, L, \leq \rangle$  satisfies properties (H1) and (H2).

#### 4.1. A hereditarily GO-space satisfies c.c.c. property

**Lemma 4.1.** *Assume that  $\langle X, \tau, L, \leq \rangle$  is a 0-dimensional hereditarily GO-structure. Then*

- (1)  *$X$  satisfies c.c.c. property.*
- (2) *The linear orderings  $\omega_1$  and  $\omega_1^*$  are not order-embeddable in  $X$ .*

*Proof.* (1) Since  $\langle X, \tau \rangle$  is 0-dimensional, any nonempty open subset of  $\langle X, \tau \rangle$  contains a clopen convex subset of the form  $(a, b)^X := (a, b)^L \cap X$  where  $a, b \in L$ . By contradiction, assume that  $\{U_i : i \in I\}$  is an uncountable family of pairwise nonempty clopen convex subsets of  $X$ . So each  $U_i$  is of the form  $(a_i, b_i)^X$  with  $a_i, b_i \in L$ .

Fix  $i_0 \in I$ . Let  $U := \bigcup \{U_i : i \in I \setminus \{i_0\}\}$  and  $\sim$  be the equivalence relation on  $X$  defined as follows:  $x \sim y$  if and only if  $x, y \in Y \setminus U$  or there is  $i \in I$  such that  $x, y \in U_i$ . Denote by  $X'$  the set  $X/\sim$  and by  $f : X \rightarrow X'$  the quotient map. We endow  $X'$  with the quotient topology  $\tau'$  on  $X'$ . So  $V' \in \tau'$  if and only if  $f^{-1}[V'] \in \tau$ . In particular,  $u_i := f[U_i]$  is an isolated point in  $X'$  for any  $i \in I \setminus \{i_0\}$ . Setting  $u = f[X \setminus U]$  we have  $X' = \{u\} \cup \bigcup \{u_i : i \in I \setminus \{i_0\}\}$  and the set

$$\{\{u_i\} : i \in I \setminus \{i_0\}\} \cup \{X' \setminus \{u_i\} : i \in I \setminus \{i_0\}\}$$

is a subbase of a topology  $\tau''$  on  $X'$  satisfying:

- (1)  $\tau'' \subseteq \tau'$  and thus  $f : \langle X, \tau \rangle \rightarrow \langle X', \tau'' \rangle$  is continuous,
- (2)  $\langle X', \tau'' \rangle$  is compact (this follow from the definition of  $\tau''$ ), and
- (3)  $\langle X', \tau'' \rangle$  is homeomorphic to the one-point compactification of the uncountable discrete set. This is so because  $u$  is the unique accumulation point of  $\langle X', \tau'' \rangle$ .

We show that  $\langle X', \tau'' \rangle$  is not a GO-space. By contradiction, suppose that  $\langle X', \tau'', L', \leq' \rangle$  is a GO-structure. Since  $\langle X', \tau'' \rangle$  is compact, by Proposition 3.6(1),  $L' = X'$  and  $\tau'' = \lambda' := \lambda^{\leq'}$ . So it suffices to prove that

- (4)  $\langle X', \tau'' \rangle := \langle X', \lambda' \rangle$  is not a LOTS.

Assume that  $\langle L', \leq' \rangle$  is a chain. Hence, for instance,  $(-\infty, u)^{L'}$  is uncountable. Consider any  $y \in (-\infty, u)$  such that  $(-\infty, y)^{L'}$  is infinite. By the definition,  $(-\infty, y)^{L'}$  is infinite, discrete, closed and thus compact, that contradicts the fact that  $u \notin (-\infty, y]^{L'}$ . We have proved that  $X$  satisfies c.c.c. property.

(2) follows from Part (1). □

Now remind the classical result which is due to Mazurkiewicz and Sierpiński (for example, see [8, Theorem 17.11], [13, Ch. 2, Theorem 8.6.10]).

**Lemma 4.2.** *Every topologically scattered compact and countable space is homeomorphic to a countable and successor ordinal space.*



#### 4.2. A hereditarily GO-space has no countable closed and relatively discrete subsets

**Lemma 4.3.** *Let  $M$  be an order-scattered LOTS. If  $M$  contains a closed and countable relatively discrete subset  $D$ , then  $D$  is a continuous image of  $M$ .*

*Proof.* First we introduce a new definition. Let  $Y$  be a topological space. For a family  $\mathcal{V}$  of pairwise disjoint subsets of  $Y$ , we denote by  $\text{acc}(\mathcal{V})$  the set of *accumulation points* of  $\mathcal{V}$ . By definition,  $x \in \text{acc}(\mathcal{V})$  if and only if for every neighborhood  $W$  of  $x$  the set  $\{V \in \mathcal{V} : V \cap W \neq \emptyset\}$  is infinite. So if  $Z \subseteq Y$ ,  $\text{acc}(Z) = \text{acc}(\{\{z\} : z \in Z\})$ . We say that a subset  $D$  of a space  $Y$  is *strongly discrete* whenever  $\mathcal{U}_D$  satisfies  $\text{acc}(D) = \text{acc}(\mathcal{U}_D)$ . The next result is well-known. For completeness we recall its proof.

*Fact 1.* *Let  $M$  be a 0-dimensional LOTS and  $D \subseteq M$ . If  $D$  is relatively discrete then  $D$  is strongly discrete.*

*Proof.* For  $d \in D$  let  $U_d := (a_d, b_d)^L$  be a clopen convex set such that  $D \cap U_d = \{d\}$ . Note the following property (\*): if  $d < d'$  then  $x < x'$  for every  $x \in U_d$  and  $x' \in U_{d'}$ . We set  $\mathcal{U}_D = \{U_d : d \in D\}$ . Obviously,  $\text{acc}(D) \subseteq \text{acc}(\mathcal{U}_D)$ . Conversely, let  $x \in \text{acc}(\mathcal{U}_D)$  and  $V$  be a neighborhood of  $x$ . We may assume that  $V$  is of the form  $(a, b)$  with  $a < b$  in  $M$ . Therefore, by (\*), there are infinitely many  $d_i$ 's such that  $d_i \in U_{d_i} \subseteq V$ , and thus  $x \in \text{acc}(\mathcal{U}_D) \subseteq \text{acc}(D)$ .  $\square$

Since  $\langle M, \leq^M \rangle$  is an order-scattered LOTS, by Proposition 3.2(2),  $\langle M, \lambda^M \rangle$  is 0-dimensional. By Fact 1, let  $\mathcal{U}_D := \{U_d : d \in D\}$  be a family of clopen convex subsets of  $M$  such that  $U_d \cap D = \{d\}$  for  $d \in D$  and  $\text{acc}(D) = \text{acc}(\mathcal{U}_D)$ . Since  $D$  is closed,  $\text{acc}(D) = \emptyset$  and thus  $\bigcup \mathcal{U}_D$  is a clopen subset of  $M$ .

Let  $d_0 \in D$  be fixed and  $\mathcal{U}_{D \setminus \{d_0\}} = \{U_d : d \in D \setminus \{d_0\}\}$ . So  $\bigcup \mathcal{U}_{D \setminus \{d_0\}} = \bigcup \mathcal{U}_D \setminus U_{d_0}$  is a clopen subset of  $M$ . Let  $\approx$  be the equivalence relation on  $M$  defined by  $x \approx y$  whenever  $x, y \in U_d$  for some  $d \in D \setminus \{d_0\}$ , or  $x, y \in M \setminus \bigcup \mathcal{U}_{D \setminus \{d_0\}}$ . For each  $x \in M$  there is a unique  $f(x) \in D$  such that  $f(x) \approx x$ . It is easy to check that the mapping  $f : M \rightarrow D$  is onto and that  $f$  is continuous: this is so, because  $f^{-1}(d)$  is clopen in  $M$  for any  $d \in D$ .  $\square$

Our next result, which is apparently well-known, strengthens Fact 6 from the proof of Corollary 1.4. Consider again  $\omega$  as a discrete space. Let  $\mathcal{U}$  on  $\omega$  be a non-principal ultrafilter on  $\omega$  and let  $*$  be a new element with  $*$   $\notin \omega$ . The space  $\mathbb{N}^* := \omega \cup \{*\}$  is equipped with the topology induced from the Stone-Ćech compactification  $\beta\mathbb{N}$ .

**Lemma 4.4.** *The countable space  $\mathbb{N}^*$  is not a GO-space.*

*Proof.* Recall that the character  $\chi(X)$  of the infinite topological space  $X$  is the supremum of cardinalities of minimal local neighborhood bases of all points in  $X$ .

*Fact 1.* *Let  $\langle X, \tau^X, L, \lambda^L \rangle$  be a GO-structure satisfying (H1) and (H2). If  $X$  is countable then  $\chi(\langle X, \tau^X \rangle) = \aleph_0$ .*

*Proof.* Since  $X$  is a countable subset of  $L$  and  $X$  is topologically dense in  $L$ , the chain  $L$  is order-embeddable in the segment  $[0, 1]$  of  $\mathbb{R}$ . Since  $\mathbb{Q} \cap [0, 1] \subseteq [0, 1]$  we have  $\chi([0, 1]) = \aleph_0$ . So,  $\aleph_0 \leq \chi(X) \leq \chi(L) \leq \chi([0, 1]) = \aleph_0$ .  $\square$

Now it suffices to remind a well-known fact that  $\chi(\mathbb{N}^*) > \aleph_0$  (see [4, 3.6.17]).

Therefore,  $\mathbb{N}^*$  is not a GO-space.  $\square$

As a consequence of the above results 4.3 and 4.4, we have:

**Corollary 4.5.** *Let  $X$  be a hereditarily GO-space. Then  $X$  does not contain an infinite countable relatively discrete closed subset.*  $\square$

#### 4.3. A hereditarily GO-space comes from a scattered linear ordering

Let GO-structure  $\langle X, \tau^X, L, \leq^L \rangle$  satisfies the conditions:

- (H1)  $X$  is topologically dense in  $\langle L, \lambda^L \rangle$ .
- (H2)  $\langle L, \leq^L \rangle$  is complete, meaning that  $\langle L, \lambda^L \rangle$  is a compact LOTS.

Hence, by Proposition 3.2(1)(b),

- (H3) For any  $u < v$  in  $L$ :  $[u, v]^L$  is order-scattered if and only if  $[u, v]^L$  is topologically-scattered.

Let  $\langle X, \tau \rangle$  be a hereditarily GO-space, meaning that  $\langle X, \tau^X, L, \leq^L \rangle$  is a hereditarily GO-structure satisfying (H1)–(H3). For simplicity denote the space  $\langle X, \tau^X \rangle$  by  $X$ . We shall show that  $\langle L, \leq^L \rangle$  is order-scattered (Proposition 4.8), and thus, by Proposition 3.2(1)(b),  $\langle L, \lambda^L \rangle$  is topologically-scattered. Therefore  $X$ , as subset of  $L$ , is also topologically-scattered.

Let  $\equiv^L$  be the equivalence relation on  $L$  defined as follows. For  $x, y \in L$ , we set  $x \equiv^L y$  if  $x \leq y$  and  $[x, y]^L$  is an order-scattered subset of  $L$ , or  $y \leq x$  and  $[y, x]^L$  is an order-scattered subset of  $L$ . Note that, by (H3), in the definition of  $\equiv^L$  we have:  $[u, v]^L$  is order-scattered if and only if  $[u, v]^L$  is topologically-scattered.

Now, each equivalence class is an order-scattered and convex subset of the LOTS  $\langle L, \leq \rangle$  and each equivalence class is closed in  $\langle L, \lambda \rangle$ . ( $\equiv^L$  is standard (see the proof of Theorem 19.26, in [9])).

We denote  $L/\equiv^L$  by  $L_1$  and by  $\pi : L \rightarrow L_1$  the projection map. So  $\pi$  is increasing and thus  $\pi$  induces a linear order  $\leq_1$  on  $L_1$ .

**Lemma 4.6.** *The linear ordering  $\langle L_1, \leq_1 \rangle$  has the following properties.*

- (1)  $L_1$  is a complete linear ordering and the quotient topology on  $L_1$  is the order topology  $\lambda_1 := \lambda^{\leq L_1}$ .
- (2)  $L_1$  is order-dense, i.e.  $L_1$  has no consecutive elements.
- (3)  $L_1$  is a compact and dense-in-itself space.
- (4)  $L_1$  is a connected space.

*Proof.* (1) This part follows from the fact that  $L$  is complete and  $\pi$  is increasing and onto.

(2)–(3) Notice first that  $\langle L_1, \leq_1 \rangle$  is a complete chain, and thus  $\langle L_1, \lambda_1 \rangle$  is compact. Secondly, there are no consecutive  $\equiv^L$ -classes in  $L$ , this is so because the union of two consecutive  $\equiv^L$ -classes is an  $\equiv^L$ -class. Hence  $L_1$  has no consecutive elements.

Therefore,  $\langle L_1, \leq_1 \rangle$  is a dense chain and thus  $L_1$  is dense-in-itself.

(4) By Part (2),  $L_1$  has no consecutive elements. Also since  $L_1$  is a complete chain,  $L_1$  has no cuts. So, by Proposition 3.6,  $L_1$  is a connected.  $\square$

Now we recall that  $X \subseteq L$  and that for  $x < y$  in  $L$ :

- (1)  $[x, y]^L$  is a scattered subspace of  $\langle L, \lambda \rangle$  if and only if  $[x, y]^L$  is a scattered subchain of  $\langle L, \leq \rangle$ , and
- (2) if  $[x, y]^L$  is a scattered subspace of  $L$  then  $[x, y]^X := [x, y]^L \cap X$  is a scattered subspace of  $X$  (but not vice-versa).

Now the relation  $\equiv^L$  induces an equivalence relation  $\equiv^X$  on  $X$ , setting for  $x, y \in X$ :

$$x \equiv^X y \text{ if and only if } x \equiv^L y.$$

We denote by  $X_1$  the space  $X/\equiv^X$ .

**Lemma 4.7.** *The following hold for  $\langle X_1, \tau^{X_1}, L_1, \leq^{L_1} \rangle$ .*

- (1) *The continuous inclusion embedding  $X \subseteq L$  induces a continuous inclusion embedding  $X_1 \subseteq L_1$ .*
- (2)  *$X_1$  is a topologically dense subset of  $L_1$  for the topology  $\lambda_1$  (on  $L_1$ ).*
- (3)  *$X_1$  considered as subordering of  $L_1$  has no consecutive points.*
- (4)  *$\langle X_1, \tau_1, L_1, \leq_1 \rangle$  is a GO-structure.*

*Proof.* (1) Let  $\pi : L \rightarrow L_1$  be the projection map. Then  $\pi[X] := X_1 \subseteq L_1$  and the embedding  $X_1 \subseteq L_1$  is continuous.

(2) Since  $X$  is a topologically dense subset of  $L$ ,  $X_1$  is topologically dense in  $L_1$ .

(3) Since  $X_1$  is topologically dense in  $L_1$ , if  $a_1 < b_1$  are consecutive elements in  $X_1$  then  $a_1 < b_1$  are also consecutive in  $L_1$ . This contradicts Lemma 4.6(2).

(4) follows from the definitions.  $\square$

We have seen that  $\langle X_1, \tau_1, L_1, \leq_1 \rangle$  is a GO-structure with the properties (H1), (H2), and the properties (1)–(4) of Lemma 4.6 and (1)–(3) of Lemma 4.7.

**Proposition 4.8.** *Let  $\langle X, \tau, L, \leq \rangle$  be a hereditarily GO-space satisfying (H1) and (H2). Then  $\langle L, \leq \rangle$  is a scattered chain.*

*Proof.* Now, with the above notations of §4.3, we consider the GO-space  $\langle X_1, \tau^{X_1}, L_1, \leq^{L_1} \rangle$  instead of the GO-space  $\langle X, \tau^X, L, \leq^L \rangle$ . Let

$$\Gamma_1 = L_1 \setminus (X_1 \cup \{\min(L_1), \max(L_1)\}).$$

That is, “ $\Gamma_1$  is the set of cuts of the chain  $\langle X_1, \leq^{X_1} \rangle$  considered as linear subordering order of  $\langle L_1, \leq^{L_1} \rangle$ ”. An obvious characterization of the elements of  $\Gamma_1$  is stated in the following fact.

*Fact 1.* *We have that  $\gamma \in \Gamma_1$  if and only if  $\gamma$  defines two nonempty clopen sets, namely  $(-\infty, \gamma)^{X_1}$  and  $(\gamma, +\infty)^{X_1}$ , for the induced topology  $\tau^{X_1}$ .*

*Therefore  $\langle X_1, \tau^{X_1} \rangle$  is a connected space if and only if  $\Gamma_1$  is empty and  $X_1$  has no consecutive elements.*  $\square$

To prove Proposition 4.8, we distinguish two cases, and in fact we prove that  $|X_1| = 1$ . This implies that  $\langle L, \leq^L \rangle$  is order-scattered.

*Case 1.*  $\Gamma_1$  has no consecutive elements and  $\Gamma_1$  is topologically dense in  $L_1$  for the order topology  $\lambda_1$ .

So the set  $\Gamma_1$  is a dense linear order and every nonempty open convex subset of  $L_1$  contains a cut. Since  $X_1$  is topologically dense in  $L_1$ , any nonempty open convex subset of  $L_1$  contains a cut and thus, in that case,

- $\langle X_1, \tau_1 \rangle$  is 0-dimensional.

Let  $c \in \Gamma_1$ . Since  $\Gamma_1$  has no first element, let  $\langle c_\alpha : \alpha < \lambda \rangle$  be a cofinal strictly increasing sequence in  $(-\infty, c)^{X_1}$  where  $\lambda$  is an infinite regular cardinal, that is, for every  $x \in (-\infty, c)^{X_1}$  there is  $\alpha$  such that  $x \leq c_\alpha$ .

*Fact 2.*  $\lambda = \omega$ .

*Proof.* If not then  $\omega_1$  is order-embeddable in  $\Gamma_1$ . Choose  $x_\alpha \in (c_\alpha, c_{\alpha+1}) \cap X_1$  for any  $\alpha$ , then  $\langle x_\alpha : \alpha < \lambda \rangle$  is a sequence in  $X_1$ , order-isomorphic to  $\omega_1$ . Since  $\langle X_1, \tau_1 \rangle$  is 0-dimensional (but not necessarily an interval space), this contradicts Lemma 4.1(2). We have proved that  $\lambda = \omega$ .  $\square$

Keeping the same notations, we have (\*): the set  $D := \{x_\alpha : \alpha \in \omega\}$  is countable and relatively discrete. Also since  $c$  is a cut, (\*\*):  $D$  is closed in  $X_1$ . (\*) together with (\*\*) contradicts Corollary 4.5. Therefore, Case 1 does not occur.

*Case 2. Not Case 1.*

This implies that either  $\Gamma_1 := L_1 \setminus (X_1 \cup \{\min(L_1), \max(L_1)\})$  has two consecutive elements in  $L_1$  or  $\Gamma_1$  is not topologically dense in  $L_1$ . Any case, there is an infinite open interval  $(u, v)^{L_1}$  of  $L_1$  (with  $u < v$  in  $L_1$ ) that does not contain a member of  $\Gamma_1$ . Recall that we have the following properties.

- (P1) The elements of  $L_1$  are exactly the  $\equiv^L$ -classes of  $L$ , and
- (P2)  $\langle L_1, \leq_1 \rangle$  is a dense linear order.

Since  $(u, v)^{L_1}$  is infinite, we may assume that  $u, v \notin \Gamma_1$ . So, we have the additional properties.

- (P3)  $[u, v]^{L_1} \cap \Gamma_1 = \emptyset$ , and thus  $[u, v]^{L_1} = [u, v]^{X_1}$ .
- (P4)  $[u, v]^{L_1}$  is infinite.
- (P5)  $X_1$ , and thus  $[u, v]^{X_1}$ , has no consecutive elements (Lemma 4.7(3)).
- (P6) Hence, by (P3), (P5) and Proposition 3.1(2),  $[u, v]^{X_1}$  is a connected space.

Next we prove that  $|X_1| = 1$ . By contradiction, assume that  $|X_1| > 1$ . We consider the equivalence relation  $\approx$  on  $X_1$  which identifies all elements of  $(-\infty, u] \cup [v, +\infty)X$ . So  $X_1/\approx$ , denoted by  $X_2$ , is a continuous image of  $X_1$ . Also  $X_2$  is a connected space and, by (P4),  $X_2$  is an infinite continuous image of  $X$ .

*Fact 3.* Let  $\langle Y, \tau^Y, M, \leq^M \rangle$  be a GO-structure such that  $\langle Y, \tau^Y \rangle$  is connected.

Then, for every  $y \in Y$ : if  $y$  is not the minimum nor the maximum of  $Y$  (if they exist) then the subspace  $Y \setminus \{y\}$  is not a connected space.

*Proof.* By Proposition 3.6(2),  $\tau^Y = \lambda^Y$ . Let  $y \in Y$  be such that  $y$  is not the minimum nor the maximum of  $Y$ . Set  $U = (\infty, y)^M$  and  $V = (y, \infty)^M$ . By the definition,  $U$  and  $V$  are open subsets of  $M$ . Hence  $U \cap Y$  and  $V \cap Y$  define a partition of  $Y$  into two nonempty open sets of  $Y$ . We have proved Fact 3.  $\square$

We claim that

*Fact 4.* For every  $x \in X_2$ , the subspace  $X_2 \setminus \{x\}$  is connected.

*Proof.* First recall that  $X_2$  is a connected space. Also we can define  $X_2$  as follows:  $X_2$  is the quotient of  $[u, v]^{X_1}$  identifying  $u$  and  $v$ . Fact 4 follows from the claim that  $[u, v]^{X_1}$  is a connected interval subspace of  $X_1$ .  $\square$

Now, from (P6) it follows that:  $X_2$  is connected, and by Fact 4: for any  $x \in X_2$  the space  $X_2 \setminus \{x\}$  is connected. Hence, by Fact 3, the space  $X_2$  is not a GO-space. So  $X_1$  and thus  $X$  is not a GO-space.

In other words, by (P1) and (P2), if  $X$  (or equivalently  $L$ ) has more than one  $\equiv^L$ -class, then  $X_2$  is a continuous image  $X$  and  $X_2$  is not a GO-space. This contradicts the fact that  $X$  is a hereditarily GO-space. We have proved that  $|X_1| = 1$ .

Further,  $|X_1| = 1$  means that  $X_1$  consists of exactly one  $\equiv^X$ -class, or, equivalently, there is the unique  $\equiv^L$ -class in  $L$ . Since for  $x < y$  in  $X$ :  $x \equiv^L y$  if and only if  $[x, y]^L$  is a scattered chain, the chain  $L$  is scattered.  $\square$

#### 4.4. End of the proof of Main Theorem

Let  $\langle X, \tau, L, \leq \rangle$  be a hereditarily GO-space. We prove first that  $X$  is countable. By Proposition 4.8,  $\langle L, \leq \rangle$  is a scattered chain. Since  $\langle L, \lambda \rangle$  is compact and topologically-scattered, by Lemma 4.1,  $\langle X, \tau \rangle$  satisfies c.c.c. property, so  $X$  has only countably many isolated points. Denote by  $\text{Iso}(Y)$  the set of isolated points of  $Y$ . Since  $X$  is topologically dense in  $L$ , we have  $\text{Iso}(L) = \text{Iso}(X)$  and thus  $\text{Iso}(L) = \aleph_0$ . Therefore, by Proposition 3.2(3),  $L$ , and thus  $X$  is also countable.

Next, by Lemma 4.3(3), the space  $X$  does not contain a countable relatively discrete set. Since  $X$  is countable,  $X$  is closed under supremum and infimum in  $L$ , and thus  $X$ , as a linear order, is complete. We have seen that  $L$  and  $X$  are compact and countable. Finally, since  $X$  is topologically-scattered, by Lemma 4.2,  $X$  is homeomorphic to the LOTS  $\alpha + 1$  where  $\alpha$  is a countable ordinal.

We have proved Main Theorem 1.1.

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